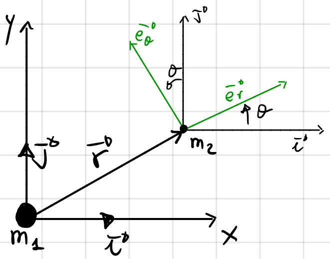
Spacecraft and missions: Solutions for problem sheet 1.

1.



$$\vec{e}_r = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{J}$$
$$\vec{e}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{J}$$

$$\frac{d\vec{e}_r}{dt} = -\sin\theta\,\dot\theta\,\vec{\imath} + \cos\theta\,\dot\theta\,\vec{J} = \dot\theta\,\vec{e}_\theta$$
$$\frac{d\vec{e}_\theta}{dt} = -\cos\theta\,\dot\theta\,\vec{\imath} - \sin\theta\,\dot\theta\,\vec{J} = -\dot\theta\,\vec{e}_r$$

NOTE $\frac{d\vec{\imath}}{dt} = \frac{d\vec{J}}{dt} = 0$!!
$\vec{\imath}, \vec{J}$ are constant unit vectors!

2. $\vec{r} = r\vec{e}_r \qquad \vec{v} = \frac{d\vec{r}}{dt} = \dot r\,\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \dot r\,\vec{e}_r + r\dot\theta\,\vec{e}_\theta$

$$\vec{h} = \vec{r}\times\vec{v} = r\vec{e}_r \times (\dot r\vec{e}_r + r\dot\theta\,\vec{e}_\theta) = \cancelto{0}{r\vec{e}_r \times \dot r\,\vec{e}_r} + r\vec{e}_r \times r\dot\theta\,\vec{e}_\theta = r^2\dot\theta\,\vec{e}_k$$

$\vec{h}$ points outside the plane

$\vec{e}_k$ unit vector ortogonal to the plane

3. $$\ddot{\vec{r}} = \frac{d\vec{v}}{dt} = \frac{d(\dot r\vec{e}_r + r\dot\theta\,\vec{e}_\theta)}{dt} = \ddot r\,\vec{e}_r + \dot r\dot\theta\,\vec{e}_\theta + \dot r\dot\theta\,\vec{e}_\theta + r\ddot\theta\,\vec{e}_\theta - r\dot\theta^2\vec{e}_r =$$

$$= (\ddot r - r\dot\theta^2)\,\vec{e}_r + (2\dot r\dot\theta + r\ddot\theta)\,\vec{e}_\theta = (\ddot r - r\dot\theta^2)\,\vec{e}_r + \frac{1}{r}\frac{d}{dt}\left(\frac{1}{2}r^2\dot\theta\right)$$

4. $$\frac{d}{dt}\vec{p} = \vec{F} \Rightarrow m\ddot{\vec{r}} = \vec{F} = -\frac{\mu m}{r^3}\vec{r}$$

$$m(\ddot{r} - r^2\dot{\theta})\vec{e_r}\cdot\vec{e_r} = -\frac{\mu m}{r^3}\overbrace{\vec{r}\cdot\vec{e_r}}^{r} \Rightarrow m(\ddot{r} - r^2\dot{\theta}) = -\frac{\mu m}{r^2}$$

$$m\frac{1}{r}\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right)\vec{e_\theta}\cdot\vec{e_\theta} = -\frac{\mu m}{r^3}\underbrace{\vec{r}\cdot\vec{e_\theta}}_{\substack{0 \\ \vec{r}\perp\vec{e_\theta}}} \Rightarrow \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \Rightarrow r^2\dot{\theta} = h = const$$